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Integral representations of weighted Beppo Levi functions

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§1. Introduction

It is well known that functions in Sobolev spaces can be represented as Bessel potentials ([17; Chapter V, Theorem 3]). In this paper we shall consider a similar problem for weighted Beppo Levi functions.

Let $1 < p < \infty$ and let w be a weight (nonnegative Lebesgue measurable function) satisfying the Muckenhoupt A_p condition:

$$(A_p) \quad \sup_Q \left(\frac{1}{|Q|} \int_Q w dx \right) \left(\frac{1}{|Q|} \int_Q w^{1/(1-p)} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes Q with sides parallel to the axes and $|Q|$ stands for the Lebesgue measure of Q (see [1]).

By A_p we denote the class of weights w satisfying (A_p) . We write

$$\|f\|_{L^p, w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}, \quad L^p(\mathbb{R}^n, w) = \{f; \|f\|_{L^p, w} < \infty\}.$$

By $BL_m(L^p(\mathbb{R}^n, w))$, $m \geq 1$, we denote the space of distributions whose partial derivatives of m -th order all belong to $L^p(\mathbb{R}^n, w)$ (see [2]).

Since $w^{1/(1-p)}$ is locally integrable by (A_p) , it follows from

Hölder's inequality and Kryloff's theorem [16; Chapitre VI, Théorème 15] that a distribution in $BL_m(L^p(\mathbb{R}^n, w))$ is a locally integrable function whenever $w \in A_p$. Therefore we call a locally integrable

function in $BL_m(L^p(\mathbb{R}^n, w))$ a Beppo Levi function of order m with

weight w . If $w = 1$, then we write simply $\|f\|_{L^p}$, $L^p(\mathbb{R}^n)$ and $BL_m(L^p(\mathbb{R}^n))$ for $\|f\|_{L^p, w}$, $L^p(\mathbb{R}^n, w)$ and $BL_m(L^p(\mathbb{R}^n, w))$, respectively. Hereafter we limit ourselves to the case $1 \leq m \leq n - 1$.

For a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ we write $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = \alpha_1! \dots \alpha_n!$ and

$$D^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}.$$

By h_m we denote the Riesz kernel $\gamma(m)^{-1} |x|^{m-n}$ with $\gamma(m) = \pi^{n/2} \Gamma(\frac{m}{2}) / \Gamma(\frac{n-m}{2})$ (cf. [17; p.117]). If $|\alpha| = m$, then $D^\alpha h_m$ is not a locally integrable function. It will be stated in Lemma 2 in §2 that $D^\alpha h_m$ is the sum of a principal value distribution S_α and a multiple $b_\alpha \delta$ of the Dirac measure δ at the origin. It is proved in [1] that this kind of distribution is related to weights w in A_p as follows:

$$(1) \quad \|(D^\alpha h_m) * g\|_{L^p, w} \leq \text{const.} \|g\|_{L^p, w} \quad \text{for } g \in L^p(\mathbb{R}^n, w).$$

Let \mathcal{D} be the space of indefinitely differentiable functions with compact support and P_{m-1} the space of all polynomials of degree smaller than or equal to $m - 1$. Let $c_\alpha = (-1)^m m! / \alpha!$. Mizuta proved

Theorem A ([8; Theorem 5.2]). Let $2m < n$ and let $f \in BL_m(L^p(\mathbb{R}^n))$. If

$$(2) \quad \text{there is a sequence } \{\psi_j\}_j \subset \mathcal{D} \text{ such that } D^\alpha \psi_j \rightarrow D^\alpha f \text{ in } L^p(\mathbb{R}^n) \\ \text{for } |\alpha| = m,$$

and $g = \sum_{|\alpha|=m} c_\alpha (D^\alpha h_m) * D^\alpha f$ satisfies

$$(3) \quad \int_{\mathbb{R}^n} (1 + |x|)^{m-n} |g(x)| dx < \infty,$$

then $f = h_m * g + P$ a.e. on R^n with some $P \in P_{m-1}$.

We shall show that assumption (2) is superfluous and the theorem extends to the case when $1 \leq m \leq n - 1$ and $f \in BL_m(L^p(R^n, w))$ with general $w \in A_p$. More precisely, we shall prove

Theorem 1. Let $w \in A_p$. Suppose that $f \in BL_m(L^p(R^n, w))$ and $g = \sum_{|\alpha|=m} c_\alpha (D^\alpha h_m) * D^\alpha f$. If g satisfies (3), then $f = h_m * g + P$ a.e. on R^n with some $P \in P_{m-1}$. Moreover this representation is unique in the sense that if $f = h_m * \mu + P'$ a.e. on R^n , where $P' \in P_{m-1}$ and μ is a signed measure such that

$$(4) \quad \int_{R^n} (1 + |x|)^{m-n} d|\mu|(x) < \infty,$$

then μ is absolutely continuous, $d\mu = gdx$ and $P' = P$.

Ohtsuka [13] proved Theorem 1 for $m = 1$ and $w = 1$ by using extremal length (see also [12] for the definition and the properties of extremal length). In case $m > 1$, however, the theory of extremal length is not applicable to $BL_m(L^p(R^n, w))$, so our argument will depend on the general theory of distributions and singular integrals (see [1], [11], [16] and [17]).

Let $\omega_{n-1} = 2\pi^{n/2}/\Gamma(\frac{n}{2})$ be the surface area of the unit sphere in R^n and let $a_\alpha = m/(\alpha! \omega_{n-1})$. Mizuta proved

Theorem B ([8; Theorem 3.1]). Let $f \in BL_m(L^p(R^n))$ satisfy (2).

If

$$(5) \quad \int_{R^n} (1 + |x|)^{m-n} |D^\alpha f(x)| dx < \infty \quad \text{for any } \alpha \text{ with } |\alpha| = m,$$

then

$$(6) \quad f(x) = \sum_{|\alpha|=m} a_\alpha \int_{\mathbb{R}^n} \frac{(x-y)^\alpha D^\alpha f(y)}{|x-y|^n} dy + P(x) \quad \text{a.e. on } \mathbb{R}^n,$$

where $P \in P_{m-1}$.

In case $m = 1$ Ohtsuka [13; Theorem 29] proved that (2) can be dropped. We shall extend Ohtsuka's result to higher order Beppo Levi functions with weight w in A_p .

Theorem 2. Let $w \in A_p$ and let $f \in BL_m(L^p(\mathbb{R}^n, w))$. If f satisfies (5), then (6) holds.

It is easy to see that $w(x) = (1 + |x|)^{rp}$ belongs to A_p if and only if $-n < rp < n(p-1)$. Hence this theorem includes Kurokawa [4; Theorem 2.6].

In case $g = \sum_{|\alpha|=m} c_\alpha (D^\alpha h_m) * D^\alpha f$ does not satisfy (3), the weighted Beppo Levi function f cannot be represented as the sum of a Riesz potential and a polynomial. However, a certain modification of the Riesz kernel (cf. [3; Chapter IV]) will enable us to represent f as the sum of a modified Riesz potential and a polynomial, and to show

Theorem 3 (cf. [14], [4; Theorem 3.2]). Let $w \in A_p$. If $f \in BL_m(L^p(\mathbb{R}^n, w))$, then there is a sequence $\{\psi_j\}_j \subset \mathcal{D}$ such that

$$\lim_{j \rightarrow \infty} \sum_{|\alpha|=m} \|D^\alpha f - D^\alpha \psi_j\|_{L^p, w} = 0.$$

In the rest of this section we deal with $w \in A_p$ for which every

$g \in L^p(\mathbb{R}^n, w)$ satisfies (3). In order to simplify the notation we denote by $A_{p,m}$ the class of all weights $w \in A_p$ such that every $g \in L^p(\mathbb{R}^n, w)$ satisfies (3). We shall show that $w \in A_{p,m}$ if and only if

$$(7) \quad \int_{\mathbb{R}^n} (1 + |x|)^{(m-n)p/(p-1)} w(x)^{1/(1-p)} dx < \infty.$$

See Theorem 7 in §5. Since $w(x) = (1 + |x|)^{rp}$ belongs to $A_{p,m}$ if and only if $m - n/p < r < n(1 - 1/p)$, it follows that $A_{p,m}$ is a proper subclass of A_p . If $w \in A_{p,m}$, then Theorem 1 gives a decomposition

$$BL_m(L^p(\mathbb{R}^n, w)) = I_m(L^p(\mathbb{R}^n, w)) \oplus P_{m-1},$$

where $I_m(L^p(\mathbb{R}^n, w)) = \{h_m * g; g \in L^p(\mathbb{R}^n, w)\}$. We shall consider a condition for $f \in BL_m(L^p(\mathbb{R}^n, w))$ to belong to $I_m(L^p(\mathbb{R}^n, w))$. For this purpose we introduce a notion which describes the behavior at ∞ of a weighted Beppo Levi function.

Definition. Let f_j and $f \in BL_m(L^p(\mathbb{R}^n, w))$. We say that f_j converges to f in the $BL_m(L^p(\mathbb{R}^n, w))$ sense if

$$\lim_{j \rightarrow \infty} \sum_{|\alpha|=m} \|D^\alpha f_j - D^\alpha f\|_{L^p, w} = 0,$$

$$\lim_{j \rightarrow \infty} f_j = f \text{ a.e. on } \mathbb{R}^n.$$

We say that f vanishes at ∞ in the $BL_m(L^p(\mathbb{R}^n, w))$ sense if there is a sequence $\{\psi_j\}_j \subset \mathcal{D}$ converging to f in the $BL_m(L^p(\mathbb{R}^n, w))$ sense.

We shall show

Theorem 4. Let $w \in A_{p,m}$. Then $f \in BL_m(L^p(\mathbb{R}^n, w))$ belongs to

$I_m(L^p(\mathbb{R}^n, w))$ if and only if f vanishes at ∞ in the $BL_m(L^p(\mathbb{R}^n, w))$ sense.

Corollary 1. Let $w \in A_{p,m}$. If $f \in BL_m(L^p(\mathbb{R}^n, w))$ and $\lim_{|x| \rightarrow \infty} f(x) = 0$, then $f \in I_m(L^p(\mathbb{R}^n, w))$, and hence f vanishes at ∞ in the $BL_m(L^p(\mathbb{R}^n, w))$ sense.

We shall give a criterion for $f \in BL_m(L^p(\mathbb{R}^n, w))$ to vanish at ∞ in terms of the integrability of f in case $w = v^p$ is a weight introduced by Muckenhoupt and Wheeden [11].

Lemma A ([11; Theorem 4]). Let $1 < p < n/m$ and $1/p^* = 1/p - m/n$. Suppose that $v \geq 0$ satisfies

$$(8) \quad \sup_Q \left(\frac{1}{|Q|} \int_Q v^{p^*} dx \right)^{1/p^*} \left(\frac{1}{|Q|} \int_Q v^{-p'} dx \right)^{1/p'} < \infty,$$

where $p' = p/(p-1)$ and the supremum is taken over all cubes Q with sides parallel to the axes. Then

$$\|(h_m * g)v\|_{L^{p^*}} \leq \text{const.} \|gv\|_{L^p} \quad \text{for } g \in L^p(\mathbb{R}^n, v^p).$$

Obviously, Hölder's inequality yields that if v satisfies (8), then $v^p \in A_p$. Hence we can easily deduce from (1) and this lemma that $v^p \in A_{p,m}$. We shall show

Theorem 5. Let $1 < p < n/m$, $1/p^* = 1/p - m/n$ and v satisfy (8).

(i) A function f in $BL_m(L^p(\mathbb{R}^n, v^p))$ vanishes at ∞ in the

$BL_m(L^p(R^n, V^p))$ sense if and only if $f \in L^{p^*}(R^n, V^{p^*})$.

(ii) If $f \in BL_m(L^p(R^n, V^p))$ satisfies

$$\int_{R^n} |f(x)|^q V(x)^r dx < \infty,$$

for some $q > 0$ and some r , $0 < r \leq p^*$, then f vanishes at ∞ in the $BL_m(L^p(R^n, V^p))$ sense.

This theorem yields the implication

$$BL_m(L^p(R^n, V^p)) \cap \left(\bigcup_{\substack{q>0 \\ 0<r\leq p^*}} L^q(R^n, V^r) \right)$$

$$\subset BL_m(L^p(R^n, V^p)) \cap L^{p^*}(R^n, V^{p^*}) = I_m(L^p(R^n, V^p)).$$

By virtue of (1), Theorem 1 and Lemma A we readily have an improvement of [11; Theorem 9].

Corollary 2. Let m , p , p^* and V be as in Theorem 5. Then there is a positive constant C depending only on m , p and V such that

$$\|fV\|_{L^{p^*}} \leq C \sum_{|\alpha|=m} \|(D^\alpha f)V\|_{L^p}$$

for $f \in BL_m(L^p(R^n, V^p)) \cap \left(\bigcup_{\substack{q>0 \\ 0<r\leq p^*}} L^q(R^n, V^r) \right)$.

§2. Preliminaries

We collect some basic results on the theory of distributions. We shall mainly use the notation of [16]. We write

$$\langle T, \psi \rangle = T(\psi)$$

for a distribution T and a test function ψ . In order to avoid confusion, we write

$$\langle T_x, \psi \rangle$$

if ψ involves two variables x and y , and the distribution T acts on $\psi(\cdot, y)$ for each fixed y . As in [16; Chapitre VII] we define the Fourier transform of $\psi \in \mathcal{S}$ and that of $T \in \mathcal{S}'$ by

$$\mathcal{F}\psi(y) = \phi(y) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot y} \psi(x) dx,$$

$$\langle \mathcal{F}T, \psi \rangle = \langle \hat{T}, \psi \rangle = \langle T, \phi \rangle \quad \text{for } \psi \in \mathcal{S},$$

where \mathcal{S} is the space of indefinitely differentiable functions decreasing rapidly at ∞ and \mathcal{S}' is the space of tempered distributions. We note that the Fourier transform defined here corresponds to the inverse Fourier transform in [17]. By \mathcal{E}' and \mathcal{D}'_{L^p} we denote the space of distributions of compact support and that of distributions T of the form

$$T = \sum_{|\alpha| \leq k} D^\alpha f_\alpha, \quad \text{where } k \geq 0 \text{ and } f_\alpha \in L^p(\mathbb{R}^n).$$

Schwartz [16; Chapitres VI and VII] proved

Lemma B. (i) If $1 \leq p \leq q \leq \infty$, then

$$\mathcal{E}' \subset \mathcal{D}'_{L^p} \subset \mathcal{D}'_{L^q} \subset \mathcal{S}' \subset \mathcal{D}'.$$

(ii) If $0 \leq 1/r = 1/p + 1/q - 1 \leq 1$, then the convolution $S * T$ exists and belongs to \mathcal{D}'_{L^r} for $S \in \mathcal{D}'_{L^p}$ and $T \in \mathcal{D}'_{L^q}$.

(iii) If S and T belong to \mathcal{D}'_{L^2} , then $\mathcal{F}S$ and $\mathcal{F}T$ belong to $L^2_{loc}(\mathbb{R}^n)$ and $\mathcal{F}(S * T) = \mathcal{F}S \cdot \mathcal{F}T$.

We can easily give another condition for the convolution of a

function and a measure to be defined.

Lemma 1. (i) Let ℓ be a real number. Suppose that $f \in L^1_{loc}(\mathbb{R}^n)$ and $|f(x)| \leq \text{const.} |x|^\ell$ for $|x| \geq 1$. If a signed measure μ satisfies

$$\int_{\mathbb{R}^n} (1 + |x|)^\ell d|\mu|(x) < \infty,$$

then $f*\mu$ is well-defined and belongs to $L^1_{loc}(\mathbb{R}^n)$; moreover

$$D^\beta(f*\mu) = (D^\beta f)*\mu = f*(D^\beta \mu) \quad \text{for any multiindex } \beta.$$

(ii) Let $0 < m < n$. If a signed measure μ satisfies (4), then $h_m*\mu$ exists and belongs to $L^1_{loc}(\mathbb{R}^n)$. If μ does not satisfy (4), then $h_m*|\mu| \equiv \infty$ on \mathbb{R}^n .

We need several results from the theory of singular integrals. Consider the class consisting of all distributions T of the form

$$(9) \quad T = c\delta + \text{v.p.} \frac{\Omega(x)}{|x|^n},$$

$$\text{i.e., } \langle T, \psi \rangle = c\psi(0) + \lim_{\epsilon \downarrow 0} \int_{|x| > \epsilon} \frac{\Omega(x)}{|x|^n} \psi(x) dx \quad \text{for } \psi \in \mathcal{D},$$

where c is a constant; Ω is a homogeneous function of degree 0, which is indefinitely differentiable on the unit sphere and

$$\int_{|x|=1} \Omega(x) d\sigma(x) = 0.$$

Lemma C ([17; Chapter III, Theorem 6]). A distribution T in \mathcal{S}' is written as (9) if and only if the Fourier transform \hat{T} is a homogeneous function of degree 0, which is indefinitely differentiable on the unit sphere.

The Muckenhoupt A_p condition is related to distributions of the form (9) as follows:

Lemma D ([1]). Let $w \in A_p$ and let T be a distribution of the form (9). Then

$$(10) \quad \|T * g\|_{L^p, w} \leq \text{const.} \|g\|_{L^p, w} \quad \text{for } g \in L^p(\mathbb{R}^n, w).$$

From Lemmas C and D we can derive a generalization of (1).

Lemma 2 (cf. [8; §3]). Let $m \geq 1$ and $\ell \geq 0$. If $|\alpha| = m$ and $|\beta| = \ell$, then the distribution

$$T = D^\alpha \left(\frac{x^\beta}{|x|^{n-m+\ell}} \right),$$

in particular $D^\alpha h_m$, is of the form (9) and satisfies (10).

By $\mathcal{D}'_{L^p, w}$ we denote the class of distributions of the form

$$\sum_{|\alpha| \leq k} D^\alpha f_\alpha, \quad \text{where } k \geq 0 \text{ and } f_\alpha \in L^p(\mathbb{R}^n, w).$$

We shall have

Lemma 3. Let S and T be distributions of the form (9) and $w \in A_p$. Then

- (i) S and T belong to \mathcal{D}'_{L^q} for any $q > 1$.
- (ii) The convolution $S * T$ exists and is of the form (9).
- (iii) If $f \in L^p(\mathbb{R}^n, w)$, then $(S * T) * f = S * (T * f) \in L^p(\mathbb{R}^n, w)$.
- (iv) If $U \in \mathcal{D}'_{L^p, w}$, then $(S * T) * U = S * (T * U) \in \mathcal{D}'_{L^p, w}$.

It is proved in [1] that every $w \in A_p$ satisfies the Muckenhoupt A_∞ condition:

There are positive constants $C, \delta > 0$ such that given any cube Q and any measurable subset E of Q ,

$$(A_\infty) \quad \frac{w(E)}{w(Q)} \leq C \left(\frac{|E|}{|Q|} \right)^\delta, \text{ where } w(A) = \int_A w dx \text{ for } A \subset \mathbb{R}^n.$$

We shall denote by A_∞ the class of weights w satisfying (A_∞) . Then

$A_\infty = \bigcup_{p>1}^\infty A_p$ (see [1]). We collect some properties of A_p and A_∞ weights.

Lemma 4. Let w_1 and w_2 belong to A_p . Then the weights $\max\{w_1, w_2\}$, $\min\{w_1, w_2\}$ and $w_1 + w_2$ belong to A_p .

Lemma 5. Let $w \in A_\infty$. If L is a cone with vertex at the origin, then $w(L) = \infty$. Moreover, there are no nonnegative functions u and v such that

$$(11) \quad \begin{aligned} & w \leq u + v \text{ on } L, \\ & \int_L u^q dx + \int_L v^r dx < \infty \text{ for some } q, r, 0 < q \leq r \leq 1. \end{aligned}$$

If a polynomial P belongs to $L^s(\mathbb{R}^n, w)$ for some $s > 0$, then $P \equiv 0$.

Lemma 6. Let $w \in A_p$. Then

$$(i) \quad \int_{\mathbb{R}^n} (1 + |x|)^{-n} |g(x)| dx < \infty \text{ for } g \in L^p(\mathbb{R}^n, w).$$

$$(ii) \quad \int_{\mathbb{R}^n} (1 + |x|)^{-np} w(x) dx < \infty.$$

§3. Proof of Theorems 1 and 2

Proof of Theorem 1. In this proof we let α, β and γ be multiindices of length m . Take $\psi \in \mathcal{D}$ such that $\psi = 1$ on a neighborhood of the origin and let $h'_m = \psi h_m$ and $h''_m = (1-\psi)h_m$. Since $h''_m \in L^q(\mathbb{R}^n)$ for any $q > n/(n-m)$ and $h'_m \in \mathcal{E}'$, it follows from Lemma B (i) that $h_m \in \mathcal{D}'_{L^q}$ for any $q > n/(n-m)$. Since $D^\alpha h_m$ belongs to \mathcal{D}'_{L^q} for any $q > 1$ by Lemmas 2 and 3, we have from Lemma B (ii) that the convolution $h_m * D^\alpha h_m$ is well-defined and belongs to \mathcal{D}'_{L^q} for any $q > n/(n-m)$. We observe that

$$D^\beta(h_m * D^\alpha h_m) = D^\alpha h_m * D^\beta h_m = h_m * D^\alpha D^\beta h_m.$$

Noting that $D^\alpha h_m \in \mathcal{D}'_{L^2}$, we obtain from Lemma B (iii) that

$$\begin{aligned} \mathcal{F}(h_m * \sum_{|\alpha|=m} c_\alpha D^{2\alpha} h_m) &= \mathcal{F}(\sum_{|\alpha|=m} c_\alpha D^\alpha h_m * D^\alpha h_m) \\ &= \sum_{|\alpha|=m} c_\alpha \{(2\pi i x)^\alpha (2\pi |x|)^{-m}\}^2 = 1, \end{aligned}$$

because

$$(2\pi |x|)^{2m} = \mathcal{F}((-\Delta)^m) = \mathcal{F}(\sum_{|\alpha|=m} c_\alpha D^{2\alpha}) = \sum_{|\alpha|=m} c_\alpha (2\pi i x)^{2\alpha}.$$

Accordingly

$$(12) \quad h_m * \sum_{|\alpha|=m} c_\alpha D^{2\alpha} h_m = \sum_{|\alpha|=m} c_\alpha D^\alpha h_m * D^\alpha h_m = \delta.$$

By (3) and Lemma 1 we obtain that the convolution $h_m * g$ is well-defined and belongs to $L^1_{loc}(\mathbb{R}^n)$. We infer from Lemmas 2, 3 and (12) that

$$\begin{aligned} D^\beta D^\gamma (h_m * g) &= D^\beta h_m * D^\gamma g = D^\beta h_m * ((\sum_{|\alpha|=m} c_\alpha D^\alpha h_m) * D^\gamma D^\alpha f) \\ &= (D^\beta h_m * \sum_{|\alpha|=m} c_\alpha D^{2\alpha} h_m) * D^\gamma f = D^\beta \delta * D^\gamma f = D^\beta D^\gamma f. \end{aligned}$$

Since β is arbitrary, it follows that

$$D^\gamma f = D^\gamma(h_m * g) + P_\gamma \quad \text{for any } \gamma \text{ with } |\gamma| = m,$$

where $P_\gamma \in P_{m-1}$. However $D^\gamma f \in L^p(\mathbb{R}^n, w)$ and

$$D^\gamma(h_m * g) = \sum_{|\alpha|=m} c_\alpha D^\gamma h_m * (D^\alpha h_m * D^\alpha f) \in L^p(\mathbb{R}^n, w)$$

by Lemma 3, and hence P_γ must be identically 0 by Lemma 5. Since

$D^\gamma f = D^\gamma(h_m * g)$ for any γ with $|\gamma| = m$, it follows that

$$f = h_m * g + P,$$

where $P \in P_{m-1}$.

The uniqueness of the representation readily follows from the following proposition, which may be of some independent interest.

Proposition 1. Let $0 < m < n$ and let μ be a signed measure satisfying (4). If $h_m * \mu$ coincides with some polynomial P , then $\mu = 0$ and hence P must be 0.

Proof. We define a sequence of signed measures μ_j of compact support by $\mu_j(E) = \mu(\{x \in E; |x| \leq j\})$. Since

$$\left| \int_{|x| > j} \psi d\mu(x) \right| \leq \text{const.} \int_{|x| > j} (1 + |x|)^{m-n} d|\mu|(x) \quad \text{for } \psi \in \mathcal{S},$$

it follows that $\mu_j \rightarrow \mu$ in \mathcal{S}' . We claim that $h_m * \mu_j \rightarrow h_m * \mu$ in \mathcal{S}' .

Let $\psi \in \mathcal{S}$. Take $\psi \in \mathcal{D}$ such that $\psi(x) = 1$ for $|x| \leq 1$ and write

$$\psi = \psi\psi + (1-\psi)\psi = \psi_1 + \psi_2.$$

It is easy to see that $h_m * |\psi_1|(x) = O(|x|^{m-n})$ as $|x| \rightarrow \infty$. Since ψ

decreases rapidly, we have $|\psi_2(y)| \leq \text{const.} |y|^{-n-1}$. Let $|x| > 2$.

Then

$$\begin{aligned}
h_m^*|\varphi_2|(x) &\leq \text{const.} \int_{|y|>1} |x-y|^{m-n} |y|^{-n-1} dy \\
&= \text{const.} |x|^{m-n-1} \left\{ \int_{|x|^{-1} < |z| < 2^{-1}} + \int_{|z|>2^{-1}} \left| \frac{x}{|x|} - z \right|^{m-n} |z|^{-n-1} dz \right\}.
\end{aligned}$$

We see that the first integral is not greater than

$$\text{const.} \int_{|x|^{-1}}^{2^{-1}} t^{-2} dt \leq \text{const.} |x|,$$

and that the second integral is a finite value independent of x .

Therefore $h_m^*|\varphi_2|(x) = O(|x|^{m-n})$ as $|x| \rightarrow \infty$. Accordingly

$$\left| \int \varphi(h_m^*(\mu - \mu_j)) dx \right| \leq \text{const.} \int_{|x|>j} |x|^{m-n} d|\mu|(x) \rightarrow 0$$

by (4) and Fubini's theorem. Thus $h_m^*\mu_j \rightarrow h_m^*\mu$ in \mathcal{S}' .

Now we see that $\mathcal{F}(h_m^*\mu_j) = (2\pi|x|)^{-m} \hat{\mu}_j \in L_{loc}^1(\mathbb{R}^n)$. In fact, since $\mu_j \in \mathcal{E}' \subset \mathcal{D}'_{L^q}$ for any $q > 1$, it follows that $h_{m/2}$ and $h_{m/2}^*\mu_j$ belong to \mathcal{D}'_{L^2} , and from Riesz's composition formula that $h_m^*\mu_j = h_{m/2}^*(h_{m/2}^*\mu_j)$. We infer from Lemma B (iii) that

$$\mathcal{F}(h_m^*\mu_j) = \mathcal{F}(h_{m/2}^*(h_{m/2}^*\mu_j)) = \hat{h}_{m/2} \cdot \hat{h}_{m/2} \cdot \hat{\mu}_j = (2\pi|x|)^{-m} \hat{\mu}_j.$$

Since the total variation of μ_j is finite, it follows that $\hat{\mu}_j$ is a bounded function, so that $(2\pi|x|)^{-m} \hat{\mu}_j \in L_{loc}^1(\mathbb{R}^n)$.

Noting that $\mu_j \rightarrow \mu$ and $h_m^*\mu_j \rightarrow h_m^*\mu = P$ in \mathcal{S}' , we obtain that

$$\hat{\mu}_j \rightarrow \hat{\mu} \quad \text{and} \quad (2\pi|x|)^{-m} \hat{\mu}_j \rightarrow \mathcal{F}(P) = P\left(\frac{-1}{2\pi i} \frac{\partial}{\partial x}\right) \delta \quad \text{in } \mathcal{S}'.$$

For any $\varphi \in \mathcal{S}$ vanishing on a neighborhood of the origin we have $\psi = (2\pi|x|)^m \varphi(x) \in \mathcal{S}$ and

$$\begin{aligned}
\langle \hat{\mu}, \varphi \rangle &= \lim_{j \rightarrow \infty} \langle \hat{\mu}_j, \varphi \rangle = \lim_{j \rightarrow \infty} \int \hat{\mu}_j(x) \varphi(x) dx \\
&= \lim_{j \rightarrow \infty} \int (2\pi|x|)^{-m} \hat{\mu}_j(x) \psi(x) dx = \langle P\left(\frac{-1}{2\pi i} \frac{\partial}{\partial x}\right) \delta, \psi \rangle = 0.
\end{aligned}$$

This implies that $\hat{\mu}$ is supported on $\{0\}$. Hence we can write

$$\hat{\mu} = P' \left(\frac{-1}{2\pi i} \frac{\partial}{\partial x} \right) \delta$$

with some polynomial P' . By the inverse Fourier transform we have $\mu = P'$, i.e., μ is absolutely continuous and $d\mu = P'dx$. Since μ satisfies (4), it follows that $(1 + |x|)^{m-n}P'(x)$ is integrable, so that P' must be identically zero. Hence $\mu = 0$ and $P = 0$.

Remark 1. The above proof works even if m is not an integer. In case m is an integer, $d\mu = gdx$, $g \in L^p(\mathbb{R}^n, w)$ and $P \in P_{m-1}$, it is possible to give a simple proof. In fact by (12) and Lemma 3

$$g = \sum_{|\alpha|=m} c_\alpha D^\alpha h_m * D^\alpha (h_m * g) = \sum_{|\alpha|=m} c_\alpha D^\alpha h_m * D^\alpha P = 0.$$

Proof of Theorem 2. By using polar coordinates and integration by parts, we can prove

$$(13) \quad \sum_{|\alpha|=m} a_\alpha D^\alpha \left(\frac{x^\alpha}{|x|^n} \right) = \delta$$

(see [15; Lemma 6.2]). Let $|\beta| = |\gamma| = m$. Applying Lemma 1 to $\ell = m - n$, $f = x^\alpha / |x|^n$ and $d\mu = D^\gamma f dx$, we obtain that

$$\left(\frac{x^\alpha}{|x|^n} \right) * D^\gamma f \in L^1_{loc}(\mathbb{R}^n) \text{ for each } \alpha \text{ and } \gamma.$$

We infer from Lemma 2 and (13) that

$$\begin{aligned} D^\beta D^\gamma \left(\sum_{|\alpha|=m} a_\alpha \left(\frac{x^\alpha}{|x|^n} \right) * D^\alpha f \right) &= \sum_{|\alpha|=m} a_\alpha D^\beta \left(\frac{x^\alpha}{|x|^n} \right) * D^\alpha D^\gamma f \\ &= \sum_{|\alpha|=m} a_\alpha D^\beta D^\alpha \left(\frac{x^\alpha}{|x|^n} \right) * D^\gamma f = D^\beta \left(\sum_{|\alpha|=m} a_\alpha D^\alpha \left(\frac{x^\alpha}{|x|^n} \right) \right) * D^\gamma f \\ &= D^\beta \delta * D^\gamma f = D^\beta D^\gamma f. \end{aligned}$$

Now the same argument as in the proof of Theorem 1 completes the proof.

§4. Proof of Theorem 3

Let us begin with modifying the Riesz kernel. The following technique is found in [3; Chapter IV] and [9, 10]. Observe that if $y \neq 0$, then $h_m(x - y)$ has a multiple power series expansion in x_1, x_2, \dots, x_n , convergent in a neighborhood of the origin. We write

$$h_m(x - y) = \sum_{v=0}^{\infty} a_v(x, y),$$

where, for fixed v and $y \neq 0$, $a_v(x, y)$ is a homogeneous polynomial in x_1 to x_n of degree v and continuous in x, y jointly for $y \neq 0$ (cf. [3; Lemma 4.1]). We now set

$$k_m(x, y) = \begin{cases} h_m(x - y) & \text{if } |y| \leq 1 \\ h_m(x - y) - \sum_{v=0}^{m-1} a_v(x, y) & \text{if } |y| > 1. \end{cases}$$

Obviously $D_x^\alpha k_m(x, y) = D_x^\alpha h_m(x - y)$ for $|\alpha| \geq m$. Since

$$|k_m(x, y)| \leq \text{const.} |x|^m |y|^{-n} \quad \text{if } 2|x| \leq |y|$$

(cf. [3; Lemma 4.2]), we can easily prove from Lemma 6 (i)

Lemma 7. Let $w \in A_p$. If $g \in L^p(\mathbb{R}^n, w)$, then

$$\int_{\mathbb{R}^n} k_m(x, y) g(y) dy \in L^1_{\text{loc}}(\mathbb{R}^n),$$

$$D^\alpha \left(\int_{\mathbb{R}^n} k_m(x, y) g(y) dy \right) = (D^\alpha h_m) * g \quad \text{for } |\alpha| \geq m.$$

This lemma and the same argument as in Theorem 1 yield

Theorem 6. Let $w \in A_p$. If $f \in BL_m(L^p(\mathbb{R}^n, w))$, then

$$f = \int_{\mathbb{R}^n} k_m(x, y) g(y) dy + P, \quad g = \sum_{|\alpha|=m} c_\alpha (D^\alpha h_m)^* D^\alpha f,$$

where $P \in P_{m-1}$.

Let \mathcal{E} be the space of all indefinitely differentiable functions on \mathbb{R}^n . We show

Lemma 8. Let $f \in I_m(L^p(\mathbb{R}^n, w)) \cap \mathcal{E}$. Then for $\varepsilon > 0$ and $r > 0$ there is a function $\psi \in \mathcal{D}$ such that

$$(14) \quad \sum_{|\alpha|=m} \|D^\alpha \psi - D^\alpha f\|_{L^p, w} < \varepsilon \quad \text{and} \quad \sup_{|x| < r} |\psi(x) - f(x)| < \varepsilon.$$

Proof. First we treat the case when $f = h_m^* g$ with $g \in \mathcal{D}$. Let $R > r$ and $\text{supp } g \in \{y; |y| < R\}$. Take $\psi \in \mathcal{D}$ such that $0 \leq \psi \leq 1$ and $\psi(x) = 1$ for $|x| < 3R$ and put $\psi_j(x) = \psi(x/j)$. We observe that

$$(15) \quad 0 \leq \psi_j \leq 1, \quad \psi_j(x) = 1 \quad \text{for } |x| < 3Rj,$$

$$\sum_{k=0}^m \sum_{|\alpha|=k} \sup(|x|^k |D^\alpha \psi_j(x)|) = \sum_{k=0}^m \sum_{|\alpha|=k} \sup(|x|^k |D^\alpha \psi(x)|) < \infty.$$

Let $h_{m,j}(x) = \psi_j(x) h_m(x)$. Then $h_{m,j}^* g \in \mathcal{D}$ and

$$\begin{aligned} h_{m,j}^* g(x) &= \int_{|y| < R} \psi_j(x-y) h_m(x-y) g(y) dy \\ &= \int_{|y| < R} h_m(x-y) g(y) dy = h_m^* g(x) \quad \text{for } |x| < 2Rj \end{aligned}$$

by (15). Let α be a multiindex of length m . We have

$$D^\alpha h_{m,j}^* g(x) = D^\alpha h_m^* g(x) \quad \text{for } |x| < 2Rj,$$

and hence

$$D^\alpha h_{m,j}^* g \rightarrow D^\alpha h_m^* g \quad \text{on } \mathbb{R}^n.$$

In view of (15) and Leibniz's formula we have

$$|D^{\alpha}h_{m,j}(x-y)| \leq \text{const.}|x|^{-n} \quad \text{for } |x| > 2R \text{ and } |y| < R,$$

and hence

$$|D^{\alpha}h_{m,j}*g(x) - D^{\alpha}h_m*g(x)| \leq \text{const.}|x|^{-n} \quad \text{for } |x| > 2R.$$

Now it follows from Lemma 6 (ii) and the dominated convergence theorem that

$$\begin{aligned} & \int_{\mathbb{R}^n} |D^{\alpha}h_{m,j}*g(x) - D^{\alpha}h_m*g(x)|^{p_w(x)} dx \\ &= \int_{|x| > 2R} |D^{\alpha}h_{m,j}*g(x) - D^{\alpha}h_m*g(x)|^{p_w(x)} dx \rightarrow 0, \end{aligned}$$

so that $D^{\alpha}(h_{m,j}*g) \rightarrow D^{\alpha}(h_m*g)$ in $L^p(\mathbb{R}^n, w)$. Therefore $\psi = h_{m,j}*g$ satisfies (14) if j is sufficiently large.

Next we consider the general case. From the uniqueness in Theorem 1 f is written as $f = h_m*g$ with $g = \sum_{|\alpha|=m} c_{\alpha}(D^{\alpha}h_m)*D^{\alpha}f \in L^p(\mathbb{R}^n, w) \cap \mathcal{E}$. It is easy to find $\psi \in \mathcal{D}$ such that $0 \leq \psi \leq 1$,

$$\sum_{|\alpha|=m} \|D^{\alpha}h_m*(\psi g) - D^{\alpha}f\|_{L^p, w} \leq \text{const.}\|\psi g - g\|_{L^p, w} < \varepsilon/2,$$

and

$$\sup_{|x| < R} |h_m*(\psi g)(x) - h_m*g(x)| < \varepsilon/2.$$

From the first part there is a function $\psi \in \mathcal{D}$ such that

$$\begin{aligned} & \sum_{|\alpha|=m} \|D^{\alpha}\psi - D^{\alpha}h_m*(\psi g)\|_{L^p, w} < \varepsilon/2, \\ & \sup_{|x| < R} |\psi(x) - h_m*(\psi g)(x)| < \varepsilon/2. \end{aligned}$$

This ψ satisfies (14).

Proof of Theorem 3. Let g be as in Theorem 6. It is easy to find a sequence $\{g_j\}_j \subset \mathcal{D}$ such that $\|g_j - g\|_{L^p, w} \rightarrow 0$. Since g_j has compact support, h_m*g_j exists and by Lemma 7

$$\int_{\mathbb{R}^n} k_m(x, y) g_j(y) dy = h_m * g_j + P_j$$

with some $P_j \in P_{m-1}$. Now Lemma 8 gives a sequence $\{\psi_j\}_j \subset \mathcal{D}$ such that

$$\sum_{|\alpha|=m} \|D^\alpha h_m * g_j - D^\alpha \psi_j\|_{L^p, w} < 1/j.$$

We infer from Theorem 6, Lemmas 2 and 7 that

$$\begin{aligned} & \sum_{|\alpha|=m} \|D^\alpha f - D^\alpha \psi_j\|_{L^p, w} \\ &= \sum_{|\alpha|=m} \|D^\alpha \left(\int_{\mathbb{R}^n} k_m(x, y) g(y) dy \right) - D^\alpha \psi_j\|_{L^p, w} \\ &= \sum_{|\alpha|=m} \|(D^\alpha h_m) * g - D^\alpha \psi_j\|_{L^p, w} \\ &\leq \sum_{|\alpha|=m} \|(D^\alpha h_m) * (g - g_j)\|_{L^p, w} + 1/j \rightarrow 0. \end{aligned}$$

The theorem is proved.

§5. Proof of Theorem 4

Lemma 9. If $\psi \in \mathcal{D}$, then $\psi = h_m * g \in I_m(L^p(\mathbb{R}^n, w))$, where $g = \sum_{|\alpha|=m} c_\alpha D^\alpha h_m * D^\alpha \psi \in L^p(\mathbb{R}^n, w) \cap \mathcal{E}$.

Proof. Since $D^\alpha h_m$ is of the form (8) and $D^\alpha \psi$ has compact support, it follows that $g(x) = O(|x|^{-n})$ as $|x| \rightarrow \infty$, so that g satisfies (3) and $h_m * g = o(1)$ as $|x| \rightarrow \infty$. By Theorem 1 we have $\psi = h_m * g + P$ with some $P \in P_{m-1}$. However P must be equal to zero, for $P(x) = \psi(x) - h_m * g(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

Lemma 10. A function $f = h_m * g + P$ in $BL_m(L^p(\mathbb{R}^n, w))$ vanishes at ∞ in the $BL_m(L^p(\mathbb{R}^n, w))$ sense if and only if there is a sequence

$\{g_j\}_j \subset L^p(\mathbb{R}^n, w) \cap \mathcal{E}$ such that

$$(16) \quad \|g_j - g\|_{L^p, w} \rightarrow 0 \quad \text{and} \quad h_m * g_j \rightarrow f \text{ a.e. on } \mathbb{R}^n.$$

Proof. First suppose that $\{g_j\}_j \subset L^p(\mathbb{R}^n, w) \cap \mathcal{E}$ satisfies (16). Then by Lemma 8 there is a sequence $\{\psi_j\}_j \subset \mathcal{D}$ such that

$$\sum_{|\alpha|=m} \|D^\alpha h_m * g_j - D^\alpha \psi_j\|_{L^p, w} < 1/j,$$

$$\sup_{|x| < j} |h_m * g_j(x) - \psi_j(x)| < 1/j.$$

We easily see that ψ_j converges to f in the $BL_m(L^p(\mathbb{R}^n, w))$ sense.

Conversely suppose that $\{\psi_j\}_j \subset \mathcal{D}$ converges to f in the $BL_m(L^p(\mathbb{R}^n, w))$ sense. We infer from Lemmas 2 and 9 that $\psi_j = h_m * g_j$, $g_j \in L^p(\mathbb{R}^n, w) \cap \mathcal{E}$ and

$$\|g_j - g\|_{L^p, w} = \left\| \sum_{|\alpha|=m} c_\alpha (D^\alpha h_m)^* (D^\alpha \psi_j - D^\alpha f) \right\|_{L^p, w}$$

converges to zero. Thus $\{g_j\}_j$ satisfies (16).

For $E \subset \mathbb{R}^n$ we define a capacity $R_{m,p,w}(E)$ by

$$R_{m,p,w}(E) = \inf \{ \|g\|_{L^p, w}^p ; g \geq 0, h_m * g \geq 1 \text{ on } E \}.$$

The next theorem combines conditions (3) and (7), the capacity $R_{m,p,w}$ and the vanishing property of Beppo Levi functions.

Theorem 7. The following statements on $w \in A_p$ are equivalent:

(a) $w \in A_{p,m}$.

(b) For every $g \in L^p(\mathbb{R}^n, w)$ the convolution $h_m * g$ exists and

belongs to $L^1_{loc}(R^n)$.

(c) If $\|g_j\|_{L^p, w} \rightarrow 0$, then $h_m * g_j \rightarrow 0$ in measure on any ball.

(c') If $\|g_j\|_{L^p, w} \rightarrow 0$, then $h_m * g_j \rightarrow 0$ in measure on some ball.

(d) The constant function 1 does not vanish at ∞ in the $BL_m(L^p(R^n, w))$ sense.

(e) There is a set of positive $R_{m,p,w}$ capacity.

(f) If $|E| > 0$, then $R_{m,p,w}(E) > 0$.

(g) w satisfies (7).

Proof. The equivalence between (a) and (b) readily follows from Lemma 1. The implications (c) \Rightarrow (c') and (f) \Rightarrow (e) are obvious. We have (f) \Rightarrow (c) from [6; Theorem 4] and (g) \Rightarrow (a) from Hölder's inequality. We shall complete the proof by showing (b) \Rightarrow (f) \Rightarrow (g), (e) \Rightarrow (b) and (c') \Rightarrow (d) \Rightarrow (a).

(b) \Rightarrow (f): Suppose that there is a measurable set E such that $|E| > 0$ but $R_{m,p,w}(E) = 0$. By [6; Theorem 3] we find a nonnegative function g in $L^p(R^n, w)$ such that $h_m * g = \infty$ on E . Since $|E| > 0$, it follows that $h_m * g$ is not locally integrable, so that (b) does not hold.

(f) \Rightarrow (g): Since the unit ball B has positive capacity, it follows from [6; Theorem 14] that there exists a measure μ concentrated on B such that $\mu(B) > 0$ and $h_m * \mu \in L^{p'}(R^n, w^{1/(1-p)})$. Noting that $h_m * \mu(x) \geq \text{const.} \mu(B) h_m(x)$ for $|x| > 1$, we obtain

$$\int_{|x|>1} |x|^{(m-n)p'} w(x)^{1/(1-p)} dx < \infty,$$

which is equivalent to (7).

(e) \Rightarrow (b): If (b) does not hold, then there is a nonnegative

function g in $L^p(\mathbb{R}^n, w)$ such that $h_m * g \equiv \infty$ on \mathbb{R}^n . By definition

$$0 \leq R_{m,p,w}(E) \leq R_{m,p,w}(\mathbb{R}^n) \leq \inf_{t>0} \|tg\|_{L^p,w}^p = 0.$$

Thus (e) does not hold.

(c') \Rightarrow (d): If 1 vanishes at ∞ in the $BL_m(L^p(\mathbb{R}^n, w))$ sense, then there is a sequence $\{g_j\}_j \subset L^p(\mathbb{R}^n, w)$ such that

$$\|g_j\|_{L^p,w} \rightarrow 0 \quad \text{and} \quad h_m * g_j \rightarrow 1 \text{ a.e. on } \mathbb{R}^n$$

by Lemma 10. This contradicts (c').

(d) \Rightarrow (a): Suppose that there is a nonnegative function g in $L^p(\mathbb{R}^n, w)$ such that (3) does not hold. Mollifying g , we may assume that $g \in L^p(\mathbb{R}^n, w) \cap \mathcal{E}$. We shall prove that 1 vanishes at ∞ in the $BL_m(L^p(\mathbb{R}^n, w))$ sense. By Lemma 10 it is sufficient to show that if $\varepsilon > 0$ and $R > 0$, then there is $g_1 \in L^p(\mathbb{R}^n, w) \cap \mathcal{E}$ such that

$$\begin{aligned} |h_m * g_1(x) - 1| &< \varepsilon \text{ for } |x| < R, \\ \|g_1\|_{L^p,w} &< \varepsilon. \end{aligned}$$

Take $R_1 > R$ such that

$$1 - \varepsilon < h_m(x-y)/h_m(y) < 1 + \varepsilon \quad \text{for } |x| < R \text{ and } |y| > R_1.$$

Since (3) does not hold and $g \in L^p(\mathbb{R}^n, w)$, we find a function $\psi \in \mathcal{D}$ such that $0 \leq \psi \leq 1$, $\text{supp } \psi \subset \{y; |y| > R_1\}$ and $g_1 = \psi g$ satisfies $h_m * g_1(0) = 1$ and $\|g_1\|_{L^p,w} < \varepsilon$. We observe that

$$1 - \varepsilon < h_m * g_1(x)/h_m * g_1(0) = h_m * g_1(x) < 1 + \varepsilon \quad \text{for } |x| < R.$$

Hence g_1 has the desired property. Thus the theorem is completely proved.

Proof of Theorem 4. Suppose that $f = h_m * g \in I_m(L^p(\mathbb{R}^n, w))$.

Take a nonnegative function ψ in \mathcal{D} such that $\int \psi dx = 1$. Letting $\psi_j(x) = j^n \psi(jx)$, we observe that $g_j = g * \psi_j \in L^p(\mathbb{R}^n, w) \cap \mathcal{E}$ satisfies (16). Hence f vanishes at ∞ in the $BL_m(L^p(\mathbb{R}^n, w))$ sense by Lemma 10.

Conversely suppose that $f = h_m * g + P \in BL_m(L^p(\mathbb{R}^n, w))$ vanishes at ∞ in the $BL_m(L^p(\mathbb{R}^n, w))$ sense. Since $h_m * g$ vanishes at ∞ in the $BL_m(L^p(\mathbb{R}^n, w))$ sense from the only if part of the theorem, it follows that $P = f - h_m * g$ vanishes at ∞ in the $BL_m(L^p(\mathbb{R}^n, w))$ sense. Hence there is a sequence $\{g_j\}_j \subset L^p(\mathbb{R}^n, w)$ such that

$$\|g_j\|_{L^p, w} \rightarrow 0 \quad \text{and} \quad h_m * g_j \rightarrow P \text{ a.e. on } \mathbb{R}^n$$

by Lemma 10. On account of (c) of Theorem 7 we have $P = 0$. The proof is complete.

For the proof of Corollary 1 we prepare

Lemma 11. Let L be a cone with vertex at the origin. Then $R_{m,p,w}(L)$ is equal to 0 or ∞ ; $R_{m,p,w}(L) = 0$ if and only if $R_{m,p,w}(\mathbb{R}^n) = 0$. The constant 1 vanishes at ∞ in the $BL_m(L^p(\mathbb{R}^n, w))$ sense if and only if $R_{m,p,w}(\mathbb{R}^n) = 0$.

Proof. If $0 < R_{m,p,w}(L) < \infty$, then there would exist a nonnegative function g in $L^p(\mathbb{R}^n, w)$ satisfying (3) and $h_m * g \geq 1$ on L by Theorem 7. Since L is not m -thin at ∞ in the notation of [5], this contradicts

$$\liminf_{|x| \rightarrow \infty, x \in L} h_m * g(x) = 0$$

([5; Theorem 3.3]). By Theorem 7 we can easily prove the remainder.

Proof of Corollary 1. Suppose that $f = h_m * g + P$ and $P \neq 0$. Then we would find $\varepsilon > 0$, $R > 0$ and a cone L with vertex at the origin such that

$$h_m * |g|(x) \geq |f(x) - P(x)| \geq \varepsilon \quad \text{if } |x| \geq R \text{ and } x \in L.$$

By definition

$$R_{m,p,w}(L) \leq R_{m,p,w}(\{x; |x| < R\}) + R_{m,p,w}(\{x \in L; |x| \geq R\}) < \infty$$

and hence by Lemma 11 $R_{m,p,w}(L) = 0$. This contradicts (f) of Theorem 7.

§6. Proof of Theorem 5

Proof of Theorem 5. First suppose that f vanishes at ∞ in the $BL_m(L^p(\mathbb{R}^n, V^p))$ sense. Thus f is written as $h_m * g$ with $g \in L^p(\mathbb{R}^n, V^p)$. On account of Lemma 10 there is a sequence $\{g_j\}_j \subset L^p(\mathbb{R}^n, V^p)$ satisfying (16) with $w = V^p$. Since

$$\|(h_m * g_j)V\|_{L^{p*}} \leq \text{const.} \|g_j V\|_{L^p} \leq \text{const.}$$

by Lemma A, Fatou's lemma leads to

$$\|fV\|_{L^{p*}} \leq \liminf_{j \rightarrow \infty} \|(h_m * g_j)V\|_{L^{p*}} < \infty.$$

The if part of (i) is included in (ii). Now we shall prove (ii) by contradiction. Suppose that $f = h_m * g + P$, $g \in L^p(\mathbb{R}^n, V^p)$, $P \in P_{m-1}$ and $P \neq 0$. Then we would find $\varepsilon > 0$, $R > 0$ and a cone L with vertex at the origin such that

$$|P(x)| \geq 2\varepsilon \quad \text{for } x \in \{x \in L; |x| \geq R\}.$$

We observe that $v^{p^*} \leq u + v$ on L , where

$$u(x) = \begin{cases} v(x)^{p^*} & \text{if } |f(x)| \geq \varepsilon \\ 0 & \text{otherwise} \end{cases}, \quad v(x) = \begin{cases} v(x)^{p^*} & \text{if } |h_m * g(x)| \geq \varepsilon \text{ or } |x| \leq R \\ 0 & \text{otherwise} \end{cases}.$$

Since $v^{p^*} \in A_\infty$ and $0 < r/p^* \leq 1$,

$$\int_L u^{r/p^*} dx = \int_{\{x \in L; |f(x)| \geq \varepsilon\}} v^r dx \leq \varepsilon^{-q} \int_{\mathbb{R}^n} |f|^q v^r dx < \infty,$$

$$\begin{aligned} \int_L v dx &= \int_{\{x \in L; |h_m * g(x)| \geq \varepsilon\}} v^{p^*} dx + \int_{|x| \leq R} v^{p^*} dx \\ &\leq \varepsilon^{-p^*} \int_{\mathbb{R}^n} |h_m * g|^{p^*} v^{p^*} dx + \int_{|x| \leq R} v^{p^*} dx < \infty, \end{aligned}$$

we have a contradiction by Lemma 5. The theorem is proved.

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